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The Kähler structure of supersymmetric holographic RG flows

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Abstract: We study the metrics on the families of moduli spaces arising from probing with a brane the ten and eleven dimensional supergravity solutions corresponding to renormalisation group flows of supersymmetric large N gauge theory. In comparing the geometry to the physics of the dual gauge theory, it is important to identify appropriate coordinates, and starting with the case of $\mathrm{SU}(N)$ gauge theories flowing from $\mathcal{N}=4$ to $\mathcal{N}=1$ *via* a mass term, we demonstrate that the metric is Kähler, and solve for the Kähler potential everywhere along the flow. We show that the asymptotic form of the Kähler potential, and hence the peculiar conical form of the metric, follows from special properties of the gauge theory. Furthermore, we find the analogous Kähler structure for the $\mathcal{N}=4$ preserving Coulomb branch flows, and for an $\mathcal{N}=2$ flow. In addition, we establish similar properties for two eleven dimensional flow geometries recently presented in the literature, one of which has a deformation of the conifold as its moduli space. In all of these cases, we notice that the Kähler potential appears to satisfy a simple universal differential equation. We prove that this equation arises for all purely Coulomb branch flows dual to both ten and eleven dimensional geometries, and conjecture that the equation holds much more generally.

Keywords: M-Theory, D-branes, AdS-CFT Correspondance, Supersymmetric Effective Theories.

Contents

1. Introduction

Using a D-brane as a probe of the ten dimensional geometry of stringy duals [1, 2, 3] of gauge theories has proven to be an interesting and useful avenue of investigation.

(There are analogous statements about the use of M-branes as probes of eleven dimensional geometries representing M-ish duals of close cousins of gauge theories.) There are a number of technical reasons for this, which all follow from the fact that the low energy action on the D-brane's world volume is *directly* related to a sector of the full gauge theory at low energy, without any need for employing holographic dictionaries which are sometimes mysterious or incomplete. With due care, this means that the quantities which arise on the brane can often be interpreted or manipulated with a direct gauge theory intuition in mind, and then used to shed light on the rest of the ten dimensional geometry. In short, the probe D-brane's world-volume is an excellent testbed for finding a good framework, such as more illuminating coordinates, in which to describe the whole gauge dual.[1]

In this paper we exhibit further examples of this fact, in the context of supergravity geometries representing the renormalisation group (RG) flow [8, 9] from $\mathcal{N} = 4$ supersymmetric pure large N SU(N) theory in the ultraviolet (UV) to supersymmetric theories of various sorts, in the infrared (IR).

We will begin with the case of flows which preserve only $\mathcal{N} = 1$ supersymmetry in the dual gauge theory, and revisit a result of ours [10] for the Coulomb branch moduli space metric obtained for the flow [11, 12] to a large N version of the Leigh-Strassler point [13, 14, 15]. There were a number of interesting features of the metric which become most apparent near the origin of moduli space, where we obtained:

$$ds^2_{\mathcal{M}_{\rm IR}} = \frac{1}{8\pi^2 g^2_{\rm YM}}\left[\frac{3}{4}du^2 + u^2\left(\frac{4}{3}\sigma_3^2 + \sigma_1^2 + \sigma_2^2\right)\right]. \tag{1.1}$$

This four dimensional branch represents the allowed vevs of two complex scalar fields ϕ_1 and ϕ_2. There is a radial coordinate u and three Euler-type angular coordinates $(\varphi_1, \varphi_2, \varphi_3)$ introduced *via* σ_i, the standard left-invariant Maurer-Cartan forms satisfying $d\sigma_i = \epsilon_{ijk}\sigma_j \wedge \sigma_k$.

The sum $\sum_{i=1}^3 \sigma_i^2$ is the standard metric on a round S^3, and so there are stretched S^3's for the radial slices due to the presence of the 4/3 instead of unity. While a deformation which preserves SU$(2)_F \times$ U$(1)_R$ (the global symmetry of the gauge theory) was to be expected, the gauge theory origin of the precise number 4/3, and hence the resulting conical singularity at the origin, and other features of the metric everywhere along the flow, all deserve some explanation.

We show that the question is best postponed until one has found better coordinates for carrying out an investigation of the implicitly defined quantities of direct relevance to a gauge theory discussion. The key point is that for a low-energy sigma model, the metric on the moduli space is the quantity which controls the kinetic terms for the scalar fields. In superspace, the kinetic terms are written in terms of a

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[1] There are a number of examples in the literature, touching on many of these aspects and issues, for instance refs. [4]–[7].

single function, the Kähler potential K:

$$\mathcal{L} = \int d^4\theta\, K(\Phi^i, \Phi^{j\dagger}) - \left\{ \int d^2\theta\, W(\Phi^i) + \text{h.c.} \right\}, \tag{1.2}$$

where Φ^i are chiral superfields whose lowest components are the scalars whose vevs we are exploring, "h.c." means "hermitian conjugate', and $W(\Phi)$ is the superpotential. For studying low energy, it is enough to keep just two derivatives in our effective action and so this is the form in which we should expect to wish to write our results.[2]

Our task then turns into one of attempting to prove the existence of a Kähler potential for the probe metric. We succeed in doing this, and this is a highly non-trivial check on the consistency of the full ten (and eleven) dimensional flow geometries from the literature which we study, and of our probe computations. An amusing feature of this computation is that the flow differential equations themselves guarantee the existence of the Kähler potential.

Having shown its existence, we can proceed to answer the gauge theory questions by determining exactly how the potential is constructed out the basic fields in the theory. This only needs the asymptotic form of the potential which we can readily write down, and using this, we show that there is a simple scaling argument which accounts for the precise form of the metric in the IR.

We can go much further that that, however. We observe that there is a strikingly simple differential equation which the potential satisfies along the flow and we are able to solve for it exactly *everywhere along the flow*! Moving away from our starting example, we find similar results for a more general $\mathcal{N} = 1$ preserving flow presented in the literature recently [16], finding the same differential equation for K.

Since this relation for K is so simple, we do a survey of many other ten and eleven dimensional RG flow geometries in the literature [17, 18, 19], preserving $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supersymmetries, and we find that our equation is apparently universal. We prove directly that it always applies to the case of the ten and eleven dimensional duals of purely Coulomb-branch flows [20, 21, 22], and are led to conjecture that it is universal. Note that this includes a slight generalisation of the differential equation which applies to the case of flows from $AdS_4 \times S^7$ and $AdS_7 \times S^4$.

Since it is notoriously difficult to obtain exact $\mathcal{N} = 1$ results for the Kähler potential — such riches are usually reserved for the superpotential, where holomorphy and non-renormalisation theorems are powerful allies — we find our results heartening, since they suggest that there is much to be gained by applying these methods further in the quest to extract useful information about strongly coupled gauge theories from supergravity duals of not inconsiderable complexity.

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[2] The D-brane probe also carries a U(1) gauge field which is decoupled from the scalar fields. Its gauge coupling is given by the ten dimensional dilaton.

2. The prototype example: an $\mathcal{N}=1$ flow in $D=4$

2.1 The ten dimensional solution

The ten dimensional solutions computed in ref. [12] describing the gravity dual of $\mathcal{N}=4$ supersymmetric SU(N) Yang-Mills theory, mass deformed to $\mathcal{N}=1$ in the IR may be written as:

$$ds^2_{10} = \Omega^2 ds^2_{1,4} + ds^2_5\,, \tag{2.1}$$

for the Einstein metric, where

$$ds^2_{1,4} = e^{2A(r)}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + dr^2\,, \tag{2.2}$$

and (see also refs. [23, 13]):

$$\begin{aligned} ds^2_5 = L^2\frac{\Omega^2}{\rho^2\cosh^2\chi}\Bigg[& d\theta^2 + \rho^6\cos^2\theta\left(\frac{\cosh\chi}{\bar{X}_2}\sigma_3^2 + \frac{\sigma_1^2+\sigma_2^2}{\bar{X}_1}\right) + \\ & + \frac{\bar{X}_2\cosh\chi\sin^2\theta}{\bar{X}_1^2}\left(d\phi + \frac{\rho^6\sinh\chi\tanh\chi\cos^2\theta}{\bar{X}_2}\sigma_3\right)^2\Bigg]\,, \end{aligned} \tag{2.3}$$

with

$$\begin{aligned} \Omega^2 &= \frac{\bar{X}_1^{1/2}\cosh\chi}{\rho} \\ \bar{X}_1 &= \cos^2\theta + \rho^6\sin^2\theta \\ \bar{X}_2 &= \mathrm{sech}\chi\cos^2\theta + \rho^6\cosh\chi\sin^2\theta\,. \end{aligned} \tag{2.4}$$

The functions $\rho(r)\equiv e^{\alpha(r)}$ and $\chi(r)$ are the supergravity scalars coupling to certain operators in the dual gauge theory. There is a one-parameter family of solutions for them which gives therefore a family of supergravity solutions. Together with $A(r)$, they obey the following equations [11]:

$$\begin{aligned} \frac{d\rho}{dr} &= \frac{1}{6L}\rho^2\frac{\partial W}{\partial\rho} = \frac{1}{6L}\left(\frac{\rho^6(\cosh(2\chi)-3)+\cosh(2\chi)+1}{\rho}\right) \\ \frac{d\chi}{dr} &= \frac{1}{L}\frac{\partial W}{\partial\chi} = \frac{1}{2L}\left(\frac{(\rho^6-2)\sinh(2\chi)}{\rho^2}\right) \\ \frac{dA}{dr} &= -\frac{2}{3L}W = -\frac{1}{6L\rho^2}\left(\cosh(2\chi)(\rho^6-2)-(3\rho^6+2)\right)\,, \end{aligned} \tag{2.5}$$

for which no explicit closed form analytic solution is known. The quantity W is the supergravity superpotential:

$$W = \frac{1}{4}\rho^4(\cosh 2\chi - 3) - \frac{1}{2\rho^2}(\cosh 2\chi + 1)\,. \tag{2.6}$$

Note that the asymptotic UV ($r \to +\infty$) behaviour of the fields $\chi(r)$ and $\alpha(r) = \log(\rho(r))$ is given by [11]:

$$\chi(r) \to a_0 e^{-r/L} + \cdots \; ; \qquad \alpha(r) \to \frac{2}{3} a_0^2 \frac{r}{L} e^{-2r/L} + \frac{a_1}{\sqrt{6}} e^{-2r/L} + \cdots . \tag{2.7}$$

Notice that this limit gives $\mathrm{AdS}_5 \times S^5$ in the UV, with cosmological constant $\Lambda = -6/L^2$ where the normalisations are such that the gauge theory and string theory quantities are related to them as:

$$L = \alpha'^{1/2} (2 g_{\mathrm{YM}}^2 N)^{1/4} \; ; \qquad g_{\mathrm{YM}}^2 = 2\pi g_s \, . \tag{2.8}$$

This limit defines the SO(6) symmetric critical point of the $\mathcal{N} = 8$ supergravity scalar potential where all of the 42 scalars vanish which is dual to the $\mathcal{N} = 4$ large N SU(N) gauge theory.

Meanwhile, in the IR ($r \to -\infty$) the asymptotic behaviour is [11]:

$$\begin{aligned} &\chi(r) \to \frac{1}{2} \log 3 - b_0 e^{\lambda r/L} + \cdots \; ; \qquad \alpha(r) \to \frac{1}{6} \log 2 - \frac{\sqrt{7}-1}{6} b_0 e^{\lambda r/L} + \cdots , \\ &\text{where} \quad \lambda = \frac{2^{5/3}}{3} (\sqrt{7} - 1) , \end{aligned} \tag{2.9}$$

defining another, SU(2)× U(1) symmetric, critical point of the scalar potential [13]. It preserves only $\mathcal{N} = 2$ supersymmetry of the maximal $\mathcal{N} = 8$ for five dimensional supergravity, dual to a conformally invariant $\mathcal{N} = 1$ supersymmetric fixed point SU(N) gauge theory at large N. It has an AdS_5 part, but the transverse part of the solution has only SU(2)× U(1), which[3] corresponds to the $\mathrm{SU}(2)_F \times \mathrm{U}(1)_R$ symmetry of the theory, which has two adjoint massless flavours transforming as an $\mathrm{SU}(2)_F$ doublet.

The fields Φ and $C_{(0)}$, the ten dimensional dilaton and R-R scalar, are gathered into a complex scalar field $\lambda = C_{(0)} + i e^{-\Phi}$ on which $\mathrm{SL}(2, \mathbb{Z})$ has a natural action. This $\mathrm{SL}(2, \mathbb{Z})$ is the duality symmetry of the gauge theory in the UV (the dilaton is related to the gauge coupling g_{YM}, and the R-R scalar to the Θ-angle), and an action of it will be inherited by the gauge theory in the IR.

The non-zero parts of the two-form potential, $C_{(2)}$, and the NS-NS two-form potential $B_{(2)}$ are listed in ref. [12], but we will not need them here. It was shown in ref. [10] that the five form field strength of the R-R four form potential $C_{(4)}$, presented in ref. [12], to which the D3-brane naturally couples, may be integrated to give a closed form for the potential, the relevant part of which which we write as:[4]

$$\begin{aligned} C_{(4)} &= -\frac{4}{g_s} w(r, \theta) \, dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 \, , \\ \text{where} \quad w(r, \theta) &= \frac{e^{4A}}{8\rho^2} [\rho^6 \sin^2\theta (\cosh(2\chi) - 3) - \cos^2\theta (1 + \cosh(2\chi))] \, . \end{aligned} \tag{2.10}$$

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[3] The SU(2) is the left-invariance of the σ_i. For details of the U(1) R-symmetry see section 7.

[4] By "relevant", we mean the part which gets pulled back to the D-brane aligned parallel to the (x^0, x^1, x^2, x^3) directions.

2.2 Moduli space metric from a probe

In ref. [10] we worked in static gauge, partitioning the spacetime coordinates, x^μ, according to:[5] $x^i = \{x^0, x^1, x^2, x^3\}$, and $y^m = \{r, \theta, \phi, \varphi_1, \varphi_2, \varphi_3\}$. We aligned the brane's worldvolume along the first four spacetime directions and obtained the effective lagrangian:

$$\mathcal{L} \equiv T - V = \frac{\tau_3}{2}\Omega^2 e^{2A} G_{mn} \dot{y}^m \dot{y}^n - \tau_3 \sin^2\theta e^{4A} \rho^4 (\cosh(2\chi) - 1)\,, \tag{2.11}$$

where the G_{mn} refer to the Einstein frame metric components, and we have neglected terms higher than quadratic order in the velocities in constructing the kinetic term. The quantity

$$\tau_3 = \frac{1}{8\pi^2 g_{YM}^2} \frac{2}{\alpha'^2}\,, \tag{2.12}$$

is the D3-brane charge under the R-R four-form potential.

The moduli space all along the flow is the four dimensional space $\sin\theta = 0$. For $r \to +\infty$ it is simply the flat metric on $\mathbb{R}^4$. In the limit $r \to -\infty$, inserting the IR values of the functions (see equation (2.9)), using the relations in equations (2.8) and defining:

$$u = \frac{\rho_0 L}{\alpha'} e^{r/\ell}\,, \qquad \ell = \frac{3}{2^{5/3}} L\,, \qquad \rho_0 \equiv \rho_{\rm IR} = 2^{1/6} \tag{2.13}$$

we get equation (1.1).

At this point, as we discussed in the previous section, it is prudent to stop and see if we can demonstrate the existence of a metric in Kähler form, since this is the low energy metric for an $\mathcal{N} = 1$ supersymmetric sigma model.

2.3 The search for Kähler structure

The moduli space is parameterised by the vevs of the massless scalars, which we shall write as z_1 and z_2. The z_i transform in the fundamental of SU(2), while their complex conjugates transform in the anti-fundamental representation. The SU(2) flavour symmetry implies that the Kähler potential is a function of u^2 only where we define,

$$u^2 = z_1 \bar{z}_1 + z_2 \bar{z}_2\,, \tag{2.14}$$

and the reader should note that this is not the coordinate u we defined in the IR in the previous subsection. We shall uncover the relation between the two shortly.

Dividing the coordinates (and indices) into holomorphic and anti-holomorphic (those without and those with a bar), if the Kähler structure exists the metric is given by

$$ds^2 = g_{\mu\bar{\nu}} dz^\mu dz^{\bar{\nu}} = g_{1\bar{1}} dz_1 d\bar{z}_1 + g_{1\bar{2}} dz_1 d\bar{z}_2 + g_{2\bar{1}} dz_2 d\bar{z}_1 + g_{2\bar{2}} dz_2 d\bar{z}_2\,, \tag{2.15}$$

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[5] Recall that the φ_i are angles on the deformed S^3 of section 2.1.

where

$$g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} K(u^2) = \partial_\mu(\partial_{\bar{\nu}}(u^2)K') = \partial_\mu(\partial_{\bar{\nu}}(u^2))K' + \partial_\mu(u^2)\partial_{\bar{\nu}}(u^2)K'' , \tag{2.16}$$

where the primes denote differentiation with respect to u^2, and we have inserted our assumption about the u dependence of K. Notice that since

$$\partial_i(u^2) = \bar{z}_i \quad \text{and} \quad \bar{\partial}_i(u^2) = z_i , \tag{2.17}$$

we have,

$$\begin{aligned} g_{1\bar{1}} &= \partial_1 \bar{\partial}_1 K = K' + z_1 \bar{z}_1 K'' , \\ g_{1\bar{2}} &= \bar{z}_1 z_2 K'' , \end{aligned} \tag{2.18}$$

and so on. So some quick algebra shows that the metric can be written as

$$ds^2 = (dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2)K' + (\bar{z}_1 dz_1 + \bar{z}_2 dz_2)(z_1 d\bar{z}_1 + z_2 d\bar{z}_2)K'' . \tag{2.19}$$

Now notice that[6]

$$\begin{aligned} du &= \frac{1}{2u}(\bar{z}_1 dz_1 + \bar{z}_2 dz_2 + z_1 d\bar{z}_1 + z_2 d\bar{z}_2) \qquad \text{and} \\ u\sigma_3 &= \frac{1}{2u}(-i\bar{z}_1 dz_1 - i\bar{z}_2 dz_2 + iz_1 d\bar{z}_1 + iz_2 d\bar{z}_2) , \end{aligned} \tag{2.20}$$

which is convenient, since we can write

$$du + iu\sigma_3 = \frac{1}{u}(\bar{z}_1 dz_1 + \bar{z}_2 dz_2) \quad \text{and} \quad du - iu\sigma_3 = \frac{1}{u}(z_1 d\bar{z}_1 + z_2 d\bar{z}_2) . \tag{2.21}$$

In exchange for a bit more algebra, we arrive at the following form for our metric:

$$ds^2 = (K' + u^2 K'')du^2 + u^2(K'(\sigma_1^2 + \sigma_2^2) + (K' + u^2 K'')\sigma_3^2) . \tag{2.22}$$

2.4 Comparison with probe result

The result derived using the brane probe should be written out at this stage, to give:

$$ds^2 = \frac{\tau_3}{2}\left\{ \frac{\cosh^2\chi}{\rho^2} e^{2A} dr^2 + L^2 \rho^2 e^{2A} (\cosh^2\chi \sigma_3^2 + \sigma_1^2 + \sigma_2^2) \right\} . \tag{2.23}$$

The explicit SU(2) invariance in this equation is that of the flavour symmetry, so in order to put the metric into Kähler form we need a change of radial coordinate relating r and u.

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[6] For instance, see ref. [24, p. 377].

Comparing equations (2.22) and (2.23) we obtain three equations:

$$(K' + u^2 K'')du^2 = \frac{\tau_3}{2}\frac{\cosh^2\chi}{\rho^2}e^{2A}dr^2\,, \tag{2.24}$$

$$u^2(K' + u^2 K'') = \frac{\tau_3}{2}L^2\rho^2 e^{2A}\cosh^2\chi\,, \tag{2.25}$$

$$u^2 K' = \frac{\tau_3}{2}L^2\rho^2 e^{2A}\,. \tag{2.26}$$

Using the first two equations we find

$$dr^2 = \frac{L^2\rho^4}{u^2}du^2\,. \tag{2.27}$$

A solution is:

$$u = \frac{L}{\alpha'}e^{f(r)/L}\,, \quad \text{with} \quad \frac{df}{dr} = \frac{1}{\rho^2}\,. \tag{2.28}$$

The latter is always positive and so defines a sensible radial coordinate u.

We can now define K by the differential equation (2.26):

$$K' = \frac{dK}{d(u^2)} = \frac{\tau_3}{2}\frac{L^2\rho^2 e^{2A}}{u^2}\,, \tag{2.29}$$

and we have to check that such a K obeys equation (2.25), which can be written as:

$$u^2\frac{d}{d(u^2)}(u^2 K') = \frac{\tau_3}{2}L^2\rho^2 e^{2A}\cosh^2\chi\,. \tag{2.30}$$

From the definition of u in equation (2.28), we have that:

$$\frac{d}{d(u^2)} = \frac{L\rho^2}{2u^2}\frac{d}{dr}\,, \tag{2.31}$$

and so we need to show

$$\frac{L\rho^2}{2}\frac{d}{dr}(u^2 K') = \frac{\tau_3}{2}L^2\rho^2 e^{2A}\cosh^2\chi\,. \tag{2.32}$$

From our definition of K in equation (2.29) this amounts to requiring us to show that:

$$\frac{d}{dr}(\rho^2 e^{2A}) = \frac{2}{L}e^{2A}\cosh^2\chi\,, \tag{2.33}$$

which seems like a tall order. Amazingly, performing the derivative on the left hand side and substituting the flow equations for $\rho(r)$ and $A(r)$ listed in (2.5) gives *precisely* the result on the right. We have demonstrated the existence of the Kähler potential! In fact, using the equation (2.31) we can write an alternative form for the definition of K, to accompany (2.29), which is:

$$\frac{dK}{dr} = \tau_3 L e^{2A(r)}\,. \tag{2.34}$$

After some thought, one can readily write down an exact solution to the equation (2.34) for the Kähler potential *everywhere along the flow*. Up to additive constants, it is:

$$K = \frac{\tau_3 L^2 e^{2A}}{4}\left(\rho^2 + \frac{1}{\rho^4}\right). \tag{2.35}$$

This is a very simple and satisfying result.

We have found a (family of) Kähler manifolds with an SU(2) × U(1) holonomy. We can readily compute interesting properties of these manifolds using the Kähler potential. For example, the Ricci form's components are readily computed to be:

$$R_{\bar{\mu}\nu} = -2\partial_{\bar{\mu}}\partial_\nu \log\left[K'(K' + u^2 K'')\right], \tag{2.36}$$

which of course generically defines a non-trivial first Chern class.

2.5 A few asymptotic results

2.5.1 Large u

For large u (i.e., in the limit of large vevs), $\rho \sim 1$ so that, from equation (2.28) we have $u \sim \frac{L}{\alpha'}\exp(r/L)$, and to leading order:

$$K \sim \frac{\tau_3}{2} L^2 e^{2r/L} = \frac{1}{8\pi^2 g_{YM}^2} u^2, \tag{2.37}$$

which implies the expected flat metric:

$$ds^2 = \frac{1}{8\pi^2 g_{YM}^2}(du^2 + u^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)). \tag{2.38}$$

We can also look at next-to-leading order corrections to the Kähler potential. Recalling the asymptotic solutions for α and χ in equations (2.7) and also the flow equations (2.5) one can show:

$$A(r) \simeq \frac{r}{L} - \frac{a_0^2}{6} e^{-2r/L} + O(e^{-4r/L}), \tag{2.39}$$

so that

$$K \simeq \tau_3 L^2 \left(\frac{1}{2} e^{2r/L} - \frac{a_0^2}{3}\frac{r}{L}\right), \tag{2.40}$$

where we have now discarded terms of order $\exp(-2r/L)$ as well as constant terms. Similarly, the corresponding expression for u^2 is

$$u^2 \simeq \frac{L^2}{\alpha'^2}\left(e^{2r/L} + \frac{4a_0^2}{3}\frac{r}{L}\right). \tag{2.41}$$

Returning to the Kähler potential, we find:

$$K \simeq \frac{1}{8\pi^2 g_{YM}^2}\left[u^2 - \frac{a_0^2 L^2}{\alpha'^2}\ln\left(\frac{\alpha'^2 u^2}{L^2}\right)\right]. \tag{2.42}$$

This expression looks similar to that which one might obtain from a one-loop calculation in field theory. To compare with such a result we need to know how a_0^2 corresponds to the mass for Φ_3. To deduce this we can look at the probe result at large u more closely. The result of the probe calculation was given in equation (2.11).

To leading order, we have

$$\begin{aligned} |z_1|^2 + |z_2|^2 &= \frac{L^2}{\alpha'^2} e^{2r/L} \cos^2\theta\,, \quad \text{and} \\ |z_3|^2 &= \frac{L^2}{\alpha'^2} e^{2r/L} \sin^2\theta\,, \end{aligned} \tag{2.43}$$

and so

$$\mathcal{L} = \frac{1}{8\pi^2 g_{YM}^2} \left((|\dot z_1|^2 + |\dot z_2|^2 + |\dot z_3|^2) - \frac{4a_0^2}{L^2} |z_3|^2 \right), \tag{2.44}$$

where the asymptotic solution for α and for χ have again been used. The mass of Φ_3 is therefore

$$m_3 = \frac{2a_0}{L}\,. \tag{2.45}$$

Inserting this into the Kähler potential, we obtain

$$K \simeq \frac{1}{8\pi^2 g_{YM}^2} u^2 - \frac{N m_3^2}{16\pi^2} \ln\left(\frac{\alpha'^2 u^2}{L^2} \right), \tag{2.46}$$

which is of the form expected for the tree-level plus one-loop correction.

2.5.2 Small u

For small u, $\rho \to 2^{1/6}$ and we have

$$u \sim \frac{L}{\alpha'} \exp\left(\frac{r}{2^{1/3} L} \right)\,. \tag{2.47}$$

This gives us the Kähler potential:

$$K \sim \frac{\tau_3}{2} L^2 \frac{3}{2^{5/3}} \left(\frac{u^2 \alpha'^2}{L^2} \right)^{4/3} = \frac{1}{8\pi^2 g_{YM}^2} \frac{3}{2^{5/3}} \left(\frac{\alpha'^2}{L^2} \right)^{1/3} (u^2)^{4/3}, \tag{2.48}$$

and so the metric in the IR is:

$$ds^2 \sim \frac{1}{8\pi^2 g_{YM}^2} 2^{1/3} \left(\frac{u^2 \alpha'^2}{L^2} \right)^{1/3} \left(\frac{4}{3} du^2 + u^2 \left(\sigma_1^2 + \sigma_2^2 + \frac{4}{3} \sigma_3^2 \right) \right), \tag{2.49}$$

which can be converted to the original form (1.1) presented in ref. [10] after the redefinition $u \to u^{3/4}$ and an overall rescaling.

So now we understand that the curious form of the IR metric, noticed in ref. [10], is simply a consequence of the power, 4/3, of u^2 which appears in the Kähler potential. As we shall see in section 7, this power shall follow from a scaling argument within the field theory.

3. A more general $\mathcal{N} = 1$ flow in $D = 4$

3.1 The ten dimensional solution

This time the solution, presented in ref. [16], allows a new scalar, $\beta = \log \nu$, to vary. The flow to non-zero values of β breaks the $\mathrm{SU}(2) \times \mathrm{U}(1)$ to $\mathrm{U}(1)^2$, and β in fact controls a vev on the Coulomb branch of the $\mathcal{N} = 1$ theory which we explored previously. If instead χ was to stay zero all along the flow this is an $\mathcal{N} = 4$ Coulomb branch flow, which we will examine more closely in section 4.

The new metric will be of the same broad structure as given in equation (2.1), with equation (2.2) for one component, while the warp factor is given by [16]:

$$\Omega^2 = \cosh\chi \left((\nu^2 \cos^2\phi + \nu^{-2} \sin^2\phi) \frac{\cos^2\theta}{\rho^2} + \rho^4 \sin^2\theta \right)^{1/2} , \tag{3.1}$$

and the deformed angular metric is given as follows [16]:

$$\begin{aligned} ds_5^2 = \; & \frac{L^2}{\Omega^2} \Big[\rho^{-4} \left(\cos^2\theta + \rho^6 \, \sin^2\theta \left(\nu^{-2} \cos^2\phi + \nu^2 \, \sin^2\phi \right) \right) d\theta^2 + \\ & \quad + \rho^2 \cos^2\theta (\nu^2 \cos^2\phi + \nu^{-2} \sin^2\phi) d\phi^2 - 2\rho^2 (\nu^2 - \nu^{-2}) \times \\ & \quad \times \sin\theta \, \cos\theta \, \sin\phi \cos\phi d\theta d\phi + \rho^2 \cos^2\theta (\nu^{-2} \cos^2\phi d\varphi_1^2 + \nu^2 \sin^2\phi d\varphi_2^2) + \\ & \quad + \rho^{-4} \sin^2\theta d\varphi_3^2 \Big] + \\ & + \frac{L^2}{\Omega^6} \sinh^2\chi \cosh^2\chi (\cos^2\theta (\cos^2\phi d\varphi_1 - \sin^2\phi d\varphi_2) - \sin^2\theta d\varphi_3)^2 \, . \end{aligned} \tag{3.2}$$

The $\mathrm{U}(1)^2$ symmetry is generated by the Killing vectors $\partial/\partial\varphi_1$ and $\partial/\partial\varphi_2$. The superpotential for this flow is given by [16]:

$$W = \frac{1}{4} \rho^4 (\cosh 2\chi - 3) - \frac{1}{4\rho^2} (\nu^2 + \nu^{-2}) (\cosh 2\chi + 1) \, , \tag{3.3}$$

which generalises the superpotential in equation (2.6). The equations of motion for the supergravity fields are:

$$\begin{aligned} \frac{d\rho}{dr} &= \frac{1}{6L} \rho^2 \frac{\partial W}{\partial \rho} = \frac{1}{12L} \left(\frac{2\rho^6 (\cosh 2\chi - 3) + (\nu^2 + \nu^{-2})(\cosh 2\chi + 1)}{\rho} \right) , \\ \frac{d\nu}{dr} &= \frac{1}{2L} \nu^2 \frac{\partial W}{\partial \nu} = -\frac{1}{4L} \left(\frac{(\cosh 2\chi + 1) \nu (\nu^2 - \nu^{-2})}{\rho^2} \right) , \\ \frac{d\chi}{dr} &= \frac{1}{L} \frac{\partial W}{\partial \chi} = \frac{\sinh 2\chi}{2L} \left(\frac{\rho^6 - (\nu^2 + \nu^{-2})}{\rho^2} \right) , \\ \frac{dA}{dr} &= -\frac{2}{3L} W = -\frac{1}{6L} \left(\frac{\rho^6 (\cosh 2\chi - 3) - (\nu^2 + \nu^{-2})(\cosh 2\chi + 1)}{\rho^2} \right) . \end{aligned} \tag{3.4}$$

The authors of ref. [16] probed the metric with a D3-brane, with the following result:

$$ds^2 = \frac{1}{2} \tau_3 e^{2A} \Big[\zeta (\rho^{-2} \cosh^2\chi dr^2 + L^2 \, \rho^2 d\phi^2) + L^2 \rho^2 (\nu^{-2} \cos^2\phi d\varphi_1^2 + \nu^2 \sin^2\phi d\varphi_2^2) +$$

$$
\begin{aligned}
&+ L^2\rho^2 \sinh^2\chi\,\zeta^{-1}(\cos^2\phi\, d\varphi_1 - \sin^2\phi\, d\varphi_2)^2\Big]\,,\\
&\text{where}\quad \zeta \equiv (\nu^2\cos^2\phi + \nu^{-2}\sin^2\phi)\,.
\end{aligned} \tag{3.5}
$$

3.2 A Kähler potential

Again we wish to use $\mathcal{N}=1$ supersymmetry and the flavour and R-symmetries of the theory to choose a special set of coordinates in which the action for the brane probe can be compared to field theory expectations. In this case the SU(2)× U(1) symmetry has been broken to $\mathrm{U}(1)^2$ by giving a vev to one of the massless fields.

The $\mathrm{U}(1)^2$ symmetries are given by constant shifts in φ_1 and φ_2. We wish to find a complex structure in which this metric is Kähler and the $\mathrm{U}(1)^2$ symmetries are realised linearly. Therefore, we should choose something like

$$
z_1 = \sqrt{u(r,\phi)}\,e^{i\varphi_1}\,, \qquad z_2 = \sqrt{v(r,\phi)}\,e^{-i\varphi_2}\,, \tag{3.6}
$$

and Kähler potential

$$
K = K(z_1\bar{z}_1, z_2\bar{z}_2) = K(u,v)\,. \tag{3.7}
$$

Proceeding as before we write down the form of the metric which results from this Kähler potential. A short calculation gives

$$
\begin{aligned}
ds^2 = {} & \frac{1}{4u^2}\left(u\frac{\partial}{\partial u}\right)^2 K du^2 + \frac{1}{2uv}\left(u\frac{\partial}{\partial u}\right)\left(v\frac{\partial}{\partial v}\right) K du dv + \frac{1}{4v^2}\left(v\frac{\partial}{\partial v}\right)^2 K dv^2 + \\
& + \left(u\frac{\partial}{\partial u}\right)^2 K d\varphi_1^2 - 2\left(u\frac{\partial}{\partial u}\right)\left(v\frac{\partial}{\partial v}\right) K d\varphi_1 d\varphi_2 + \left(v\frac{\partial}{\partial v}\right)^2 K d\varphi_2^2\,.
\end{aligned} \tag{3.8}
$$

Comparison with equation (3.5) gives the following set of equations for u, v and K.

$$
\begin{aligned}
\left(u\frac{\partial}{\partial u}\right)^2 K &= \frac{\tau_3}{2} e^{2A} L^2\rho^2(\nu^{-2}\cos^2\phi + \sinh^2\chi\,\zeta^{-1}\cos^4\phi)\\
&= 4u^2 e^{2A}\zeta\left(\rho^{-2}\cosh^2\chi\left(\frac{\partial r}{\partial u}\right)^2 + L^2\rho^2\left(\frac{\partial\phi}{\partial u}\right)^2\right),\\
\left(v\frac{\partial}{\partial v}\right)^2 K &= \frac{\tau_3}{2} e^{2A} L^2\rho^2(\nu^{2}\sin^2\phi + \sinh^2\chi\,\zeta^{-1}\sin^4\phi)\\
&= 4v^2 e^{2A}\zeta\left(\rho^{-2}\cosh^2\chi\left(\frac{\partial r}{\partial v}\right)^2 + L^2\rho^2\left(\frac{\partial\phi}{\partial v}\right)^2\right),\\
\left(u\frac{\partial}{\partial u}\right)\left(v\frac{\partial}{\partial v}\right) K &= \frac{\tau_3}{2} e^{2A} L^2\rho^2\sinh^2\chi\,\zeta^{-1}\cos^2\phi\sin^2\phi\\
&= 4uv e^{2A}\zeta\left(\rho^{-2}\cosh^2\chi\left(\frac{\partial r}{\partial u}\right)\left(\frac{\partial r}{\partial v}\right) + L^2\rho^2\left(\frac{\partial\phi}{\partial u}\right)\left(\frac{\partial\phi}{\partial v}\right)\right).
\end{aligned}
$$

The solutions for u and v are

$$
u = f(r)\cos^2\phi\,, \qquad v = g(r)\sin^2\phi\,, \tag{3.9}
$$

where

$$\frac{df}{dr} = \frac{2\nu^2}{L\rho^2} f \,, \qquad \frac{dg}{dr} = \frac{2}{L\rho^2\nu^2} g \,. \tag{3.10}$$

We find an exact solution for the Kähler potential:

$$K = \frac{\tau_3}{2} L^2 e^{2A} \left(\rho^2(\nu^2 - \nu^{-2}) \sin^2\phi + \frac{1}{2}(\rho^2\nu^{-2} + \rho^{-4}) \right) + a\log(u) + b\log(v) + d \,, \tag{3.11}$$

where a, b and d are constants. As before the equations of motion (3.4) were needed in order to find a solution. The specific combinations of the equations used are rather simple and we reproduce them below:

$$\frac{d(e^{2A}\rho^2\nu^{-2})}{dr} = \frac{d(e^{2A}\rho^2\nu^2)}{dr} = \frac{2}{L} e^{2A}\cosh^2\chi \tag{3.12}$$

$$\frac{d(e^{2A}\rho^{-4})}{dr} = \frac{e^{2A}}{L}(3 - \cosh(2\chi)) \,. \tag{3.13}$$

In fact, this allows us to write down a remarkably simple solution for A as a function of ρ and ν for the case $\nu \neq 1$:

$$e^{2A} = \frac{k}{\rho^2(\nu^2 - \nu^{-2})} \,, \tag{3.14}$$

where k is a constant. Using this expression and setting $a = b = d = 0$ we can simplify the Kähler potential and we find again that it satisfies the equation (2.34)

$$K = \frac{\tau_3}{2} L^2 k \sin^2\phi + \frac{\tau_3 L^2 e^{2A}}{4} \left(\frac{\rho^2}{\nu^2} + \frac{1}{\rho^4} \right) . \tag{3.15}$$

4. Pure Coulomb branch flows

4.1 Switching off the mass in $D = 4$

In fact, it is quite interesting to study a special case of the above. Let us switch off the mass deformation by setting $\chi = 0$ everywhere. This means that we are studying a purely $\mathcal{N} = 4$ Coulomb branch deformation. If we set $\chi = 0$ then equations (3.12) and (3.13) simplify to yield the result

$$e^{2A}\rho^2\nu^2 = e^{2A}\rho^2\nu^{-2} + k = e^{2A}\rho^{-4} + l \,, \tag{4.1}$$

where k is the constant appearing in equation (3.14) and l is another integration constant. We can also write down a solution for equation (3.10) in this case:

$$f = \frac{L^2}{\alpha'^2} e^{2A}\rho^2\nu^{-2}, \qquad g = \frac{L^2}{\alpha'^2} e^{2A}\rho^2\nu^2 \,. \tag{4.2}$$

Now if we substitute into the expression (3.15) for the Kähler potential, we find

$$\begin{aligned} K &= \frac{\tau_3}{2} L^2 e^{2A} (\rho^2 \nu^2 \sin^2\phi + \rho^2 \nu^{-2} \cos^2\phi) \\ &= \frac{1}{8\pi^2 g_{YM}^2}(u+v) \\ &= \frac{1}{8\pi^2 g_{YM}^2}(z_1 \bar{z}_1 + z_2 \bar{z}_2)\,. \end{aligned} \tag{4.3}$$

This is the expected probe result [7] for a two complex dimensional subspace of the flat three complex dimensional moduli space which exists for the $\mathcal{N}=4$ theory on Coulomb branch. Notice that in the standard supergravity flow coordinates, this result is not manifest, but there is a change of coordinates (achievable with the aid of the flow equations[7]) to make it so. The ten dimensional flow metric in the new coordinates is then nothing else but a distributional D3-brane solution of ref. [25], for which the transverse space is flat space multiplied by an harmonic function.

Let us see if we can write the general result for the pure Coulomb branch flows, knowing that the underlying structure of the ten dimensional solution is so simple.

4.2 The general case of the Coulomb branch

The general $\mathcal{N}=4$ supersymmetric Coulomb branch flows are discussed in refs. [20, 21]. The five dimensional supergravity equations describing these flows were given in ref. [20] and a way to decouple these equations in order to find exact solutions was presented in ref. [21]. The ten-dimensional metric corresponds to a continuous distribution of D3-branes [25] and so must take the form

$$ds^2 = H^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H^{1/2}(dy_1^2 + dy_2^2 + \cdots + dy_6^2)\,, \tag{4.4}$$

for some harmonic function $H(y_i)$. A D3-brane probing this background has a flat metric on moduli space [7]:

$$ds^2 = \frac{\tau_3}{2}(dy_1^2 + dy_2^2 + \cdots + dy_6^2) = \frac{\tau_3}{2}(dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2 + dz_3 d\bar{z}_3)\,, \tag{4.5}$$

coming from a Kähler potential

$$K = \frac{\tau_3}{2}(y_1^2 + y_2^2 + \cdots + y_6^2)\,. \tag{4.6}$$

In ref. [21] it was shown that it is natural to write the lift solution in terms of a radial coordinate F which satisfies, in the coordinates of our discussion:

$$\frac{dF}{dr} = 2Le^{2A}\,, \tag{4.7}$$

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[7] See also refs. [6, 20, 21].

and is related to the y_i by

$$y_i = (F - b_i)^{1/2}\hat{x}_i\,, \tag{4.8}$$

where b_i are constants and $\hat{x}_i$ are coordinates on a unit S^5. Changing to F coordinates we find that the Kähler potential is given by

$$K = \frac{\tau_3}{2}\sum_i (F - b_i)\hat{x}_i{}^2 = \frac{\tau_3}{2}(F - \sum_i b_i\hat{x}_i{}^2)\,, \tag{4.9}$$

from which we can read

$$\frac{\partial K}{\partial F} = \frac{\tau_3}{2}\,, \tag{4.10}$$

which when combined with equation (4.7) results in our equation (2.34) once again. In fact it is straightforward to generalise these results to the case of M2- and M5-branes on their analogues of the Coulomb branch by adapting the results of ref. [22]. In these cases we find

$$\frac{\partial K}{\partial r} = \tau_{M2} L e^{A}\,, \tag{4.11}$$

for M2-branes and

$$\frac{\partial K}{\partial r} = \tau_{M5} L e^{4A}\,, \tag{4.12}$$

for M5-branes. We conjecture that together with the equation (2.34) for systems built of D3-branes, the natural Kähler potentials governing the metric on moduli space for all supersymmetric holographic RG flows coming from gauged supergravities, satisfy these equations. We have proven them for purely Coulomb branch flows, shown them for the Leigh-Strassler flow and a generalisation, and in the next two sections demonstrate that it is also true for another three families of examples.

5. An $\mathcal{N} = 2$ flow in $D = 4$

We now briefly revisit the ten-dimensional $\mathcal{N} = 2$ supersymmetric flow solution constructed in ref. [17], which has been studied *via* brane probing in refs. [4, 5, 6]. Again we wish to find an explicit form of the Kähler potential and check that equation (2.34) is satisfied. In this case the moduli space for a D3-brane probe is a one complex dimensional space and so it is simple to find coordinates in which the metric is Kähler, since these are just the coordinates in which the metric is conformally flat:

$$ds^2 = \partial\bar{\partial}K\, dz\, d\bar{z}\,. \tag{5.1}$$

In fact such coordinates have already been found in refs. [4, 5], and we reproduce the result here

$$ds^2 = \frac{\tau_3}{2}k^2 L^2 \frac{c}{(c+1)^2}\, dz\, d\bar{z}\,, \tag{5.2}$$

where

$$c = \cosh(2\chi) \quad \text{and} \quad z = e^{-i\phi}\sqrt{\frac{(c+1)}{(c-1)}} \,. \tag{5.3}$$

Note that this fixes the complex structure on moduli space but that in this case the flavour symmetries are insufficient to pin down a unique choice of complex coordinates. $\mathcal{N} = 2$ supersymmetry is used in ref. [4] to match the scalar kinetic term with the kinetic term of the U(1) gauge field on the brane and so fix a unique set of coordinates.[8] However, for our purposes we only need the correct complex structure and so $\mathcal{N} = 1$ arguments are enough. To proceed we set $u = z\bar{z}$ and find

$$ds^2 = \frac{d}{du}\left(u\frac{d}{du}\right) K \, dz \, d\bar{z} = \frac{\tau_3}{2} k^2 L^2 \frac{(u^2-1)}{4u^2} \, dz \, d\bar{z} \,, \tag{5.4}$$

which has solution

$$K = \frac{\tau_3}{2} k^2 L^2 \left(\frac{1}{4}(u - u^{-1}) + a\log(u) + b\right), \tag{5.5}$$

where a and b are constants. To get this expression for the Kähler potential into the form we want, we need to use the following solutions [17] for the supergravity fields A and ρ,

$$\begin{aligned} e^A &= k\frac{\rho^2}{\sinh 2\chi} \\ \rho^6 &= \cosh 2\chi + \sinh^2 2\chi \left(\gamma + \log\left[\frac{\sinh\chi}{\cosh\chi}\right]\right). \end{aligned} \tag{5.6}$$

If we choose $a = -\frac{1}{2}$ and $b = \gamma$ then the Kähler potential simplifies to

$$K = \frac{\tau_3}{2} L^2 \rho^2 e^{2A} \,, \tag{5.7}$$

which on applying the relevant supergravity equations of motion

$$\begin{aligned} \rho\frac{d\rho}{dr} &= \frac{1}{3L}\left(\frac{1}{\rho^2} - \rho^4\cosh 2\chi\right), \quad \frac{d\chi}{dr} = -\frac{1}{2L}\rho^4 \sinh 4\chi \,, \\ \frac{dA}{dr} &= \frac{2}{3L}\left(\frac{1}{\rho^2} + \frac{1}{2}\rho^4\cosh 2\chi\right), \end{aligned} \tag{5.8}$$

can be shown to satisfy equation (2.34).

6. Two $\mathcal{N} = 2$ flows in $D = 3$

Finally, we examine two non-trivial examples in eleven dimensions which appeared in the literature very recently [19]. The first is a flow from the SO(8) invariant 2+1

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[8] See also ref. [6] for further work and generalisations.

dimensional fixed point in the UV (dual to $\mathrm{AdS}_4 \times S^7$) to an SU(3) × U(1) fixed point generalising the fixed point $D = 4$ field theory of section 2. It is dual to an $\mathcal{N} = 2$ gauged supergravity critical point [26], and aspects of the dual field theory have been considered in refs. [27]. The second is closely related (from the eleven dimensional point of view), being an alternative lift of the *same* four dimensional $\mathcal{N} = 8$ supergravity fields as the first example to a different eleven dimensional completion for which the moduli space in the limit of large fields is the conifold instead of $\mathbb{R}^8$. This is achieved by replacing the stretched S^5's (which generalise the stretched S^3's in sections 2 and 3) with a space which is topologically $T^{1,1}$; the authors of ref. [19] noticed that the same gauged supergravity equations of motion result. We make a few more comments about the field theory dual of this second example towards the end of subsection 6.2.

6.1 The S^5 flow

The flow under consideration has a 3 (complex) dimensional moduli space with SU(3) symmetry and we start by finding the general form for an SU(n) invariant Kähler metric on $\mathbb{C}^n$. Let $w^1, w^2, \ldots, w^n$ be coordinates on $\mathbb{C}^n$. SU(n) invariance implies that the Kähler potential K depends only on the combination

$$q = w^1\bar{w}^1 + w^2\bar{w}^2 + \cdots + w^n\bar{w}^n \,. \tag{6.1}$$

Let us reparametrise $\mathbb{C}^n \sim \mathbb{R}^{2n} \sim \mathbb{R}^+ \times S^{2n-1}$ with coordinates $\hat{x}^1, \ldots, \hat{x}^{2n}, q$. The $\hat{x}$'s are coordinates on an S^{2n-1} of unit radius and are related to the w's by

$$w^J = \sqrt{q}(\hat{x}^{2J-1} + i\hat{x}^{2J}) \,. \tag{6.2}$$

A short calculation gives the Kähler metric in these coordinates as:

$$ds^2 = \frac{1}{4q^2}\left(q\frac{d}{dq}\right)^2 K\, dq^2 + \left(q\frac{d}{dq}\right) K\, d\hat{x}^I d\hat{x}^I + \left(q^2\frac{d^2}{dq^2}\right) K\, (\hat{x}^I J_{IJ} d\hat{x}^J)^2 \,, \tag{6.3}$$

where J is an antisymmetric matrix with $J_{12} = J_{34} = \cdots = J_{2n-1\,2n} = 1$.

Now we wish to compare this to the metric on moduli space for the flow solution in ref. [19]. The lift ansatz for the eleven dimensional metric in ref. [19] is:

$$ds^2_{11} = \Delta^{-1}(dr^2 + e^{2A(r)}(\eta_{\mu\nu}dx^\mu dx^\nu)) + \Delta^{1/2}L^2 ds^2(\rho,\chi) \,, \tag{6.4}$$

where $ds^2(\rho,\chi)$ is given in equation (4.10) of ref. [19] and we will give the restriction of it to moduli space below. From this form of the eleven dimensional metric and a short calculation to determine the location of the moduli space we can read off the metric on moduli space:

$$ds^2 = \frac{\tau_{M2}}{2} e^A(\Delta^{-3/2}dr^2 + L^2 ds^2(\rho,\chi)|_{\text{moduli}}) \,, \tag{6.5}$$

where

$$\Delta^{-3/2} = \frac{\cosh^2\chi}{\rho^2}\,, \tag{6.6}$$

on moduli space and:

$$ds^2(\rho,\chi)|_{\text{moduli}} = \rho^2 d\hat{x}^I d\hat{x}^I + \rho^2\sinh^2\chi(\hat{x}^I J_{IJ} d\hat{x}^J)^2\,,\quad I,J = 1,\ldots,3\,, \tag{6.7}$$

which defines a family of stretched[9] S^5's generalising the stretched S^3's of our $D = 4$ gauge theory result in equation (2.23).

Now we can find the complex structure and Kähler potential which produce such a metric on moduli space. Substituting equation (6.7) into (6.5) and comparing with equation (6.3), we get three equations for the two unknown functions $q(r)$ and $K(r)$:

$$\frac{1}{4q^2}\left(q\frac{d}{dq}\right)^2 K dq^2 = \frac{\tau_{M2}}{2}e^A\Delta^{-3/2}dr^2\,, \tag{6.8}$$

$$\left(q\frac{d}{dq}\right)K = \frac{\tau_{M2}}{2}e^A L^2\rho^2\,, \tag{6.9}$$

$$\left(q^2\frac{d^2}{dq^2}\right)K = \frac{\tau_{M2}}{2}e^A L^2\rho^2\sinh^2\chi\,. \tag{6.10}$$

Consistency of these equations requires that

$$\frac{d}{dr}(e^A\rho^2) = \frac{2}{L}e^A\cosh^2\chi\,, \tag{6.11}$$

which is a consequence of the supergravity equations of motion. Then the solutions for K and q are:

$$K = \frac{\tau_{M2}L^2}{4}e^A\left(\rho^2 + \frac{1}{\rho^6}\right)\,,\qquad \frac{dq}{dr} = \frac{2}{L\rho^2}q\,. \tag{6.12}$$

We observe again that, happily, this Kähler potential satisfies equation (4.11), and the contrast with equation (2.35) for the analogous four dimensional flow should be noted.

6.2 The $T^{1,1}$ flow

Interestingly, there is a second lift of the same four dimensional gauged supergravity solution which led to the eleven dimensional example of the previous section. This was also constructed in ref. [19]. The second eleven dimensional geometry is similar to the first, but with the stretched five spheres (U(1) bundles over $\mathbb{CP}^2$) replaced by stretched $T^{1,1}$'s, (U(1) bundles over $S^2\times S^2$). The details are in ref. [19] and since we shall only be interested in the moduli space of an M2-brane probe, we shall not go into details here.

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[9] Note that $d\hat{x}^I d\hat{x}^I$ is the metric on a round S^5 while $(\hat{x}^I J_{IJ} d\hat{x}^J)^2$ is the U(1) fibre in the description of S^5 as a U(1) fibre over $\mathbb{CP}^2$. So the function $\sinh^2\chi$ controls a stretching of the fibre in a precise generalisation of the case for the S^3's in equation (2.23).

The metric on moduli space for an M2-brane probing this geometry is again given by equation (6.5) except that equation (6.7) is replaced by

$$ds^2(\rho,\chi)|_{\text{moduli}} = \rho^2 ds^2_{T^{1,1}} + \rho^2 \sinh^2\chi \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 \,, \tag{6.13}$$

and the metric on $T^{1,1}$ is [28]:

$$ds^2_{T^{1,1}} = \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \frac{1}{6}(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \,. \tag{6.14}$$

In particular, the S^5's in equation (6.7) have been replaced by $T^{1,1}$'s so that for large r ($\rho \to 1$, $\chi \to 0$) the two moduli spaces approach flat $\mathbb{R}^6$, and the Ricci flat Kähler conifold, respectively.

To proceed we first note that the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ symmetry of the conifold metric is preserved for all values of r. Therefore, we need to find the general form of an $\mathrm{SU}(2)\times\mathrm{SU}(2)$ invariant Kähler metric on the conifold and compare to equations (6.5), (6.13). Such metrics have been studied in ref. [28] and we shall rederive a result of that paper below.

The conifold is a surface in $\mathbb{C}^4$ parametrised by four complex coordinates z_1, z_2, z_3 and z_4, which satisfy an equation

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \,. \tag{6.15}$$

An $\mathrm{SU}(2)\times\mathrm{SU}(2) = \mathrm{SO}(4)$ invariant metric depends only on the combination

$$p = z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 + z_4\bar{z}_4 \,. \tag{6.16}$$

Thus, to construct an $\mathrm{SU}(2)\times\mathrm{SU}(2)$ invariant metric on the conifold we can start with our equation (6.3) for an SU(4) invariant metric on $\mathbb{C}^4$ and then restrict to the conifold using equation (6.15). For convenience and in order to adjust a couple of notations we reproduce the equation for an SU(4) invariant metric on $\mathbb{C}^4$ here:

$$ds^2 = \frac{1}{4p^2}\left(p\frac{d}{dp}\right)^2 K\, dp^2 + \left(p\frac{d}{dp}\right) K\, d\hat{z}_i d\hat{\bar{z}}_i + \left(p^2\frac{d^2}{dp^2}\right) K\, |\hat{z}_i d\hat{\bar{z}}_i|^2 \,, \tag{6.17}$$

where the $\hat{z}_i$'s parametrise an S^7, i.e. $\Sigma \hat{z}_i \hat{\bar{z}}_i = 1$. In fact it will be more convenient to work in terms of a radial coordinate $q = \frac{3}{2}p^{2/3}$ for reasons which will become clear. In these coordinates the SU(4) invariant metric becomes:

$$ds^2 = \frac{1}{4q^2}\left(q\frac{d}{dq}\right)^2 K\, dq^2 + \left(q\frac{d}{dq}\right) K\, \left[\frac{2}{3} d\hat{z}_i d\hat{\bar{z}}_i - \frac{2}{9}|\hat{z}_i d\hat{\bar{z}}_i|^2\right] + \frac{4}{9}\left(q^2\frac{d^2}{dq^2}\right) K\, |\hat{z}_i d\hat{\bar{z}}_i|^2 \,. \tag{6.18}$$

Finally, we need to restrict to the conifold by applying equation (6.15) to the $\hat{z}_i$'s. If we reparametrise in terms of the coordinates[10] on $T^{1,1}$ introduced in equation (6.14),

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[10] The reader should refer to ref. [28] for the explicit form of these coordinates in terms of the $\hat{z}_i$'s.

then we find the following form for the general SU(2) × SU(2) invariant metric on the conifold:

$$ds^2 = \frac{1}{4q^2}\left(q\frac{d}{dq}\right)^2 K\, dq^2 + \left(q\frac{d}{dq}\right) K\, [ds^2_{T^{1,1}}] + \frac{1}{9}\left(q^2\frac{d^2}{dq^2}\right) \times$$
$$\times K\, (d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2\,. \qquad (6.19)$$

It is now straightforward to compare this metric with the metric on moduli space (6.5), (6.13), to extract equations for K and q. The equations which K and q satisfy are precisely (6.8), (6.9) and (6.10) with solution (6.12). It should be noted, however, that q has a very different definition than it did in the flow of the previous section.

7. Scaling dimensions and R-symmetries

The supergravity flows discussed in sections 2 and 6 interpolate between geometries of the form AdS×M, corresponding to flows between conformal field theories. We have already fixed unique coordinates on moduli space by demanding that the supersymmetry and flavour symmetries be realised linearly in the brane probe action. Now we should check whether the scaling dimensions of the chiral superfields are correctly reproduced in these coordinates.

7.1 Ten dimensional $\mathcal{N} = 1$ flow

A simple example to start with is the UV end of the flow in section 2 which is just the standard $AdS_5 \times S^5$ geometry. The AdS_5 part of the metric is

$$ds^2 = e^{2A}\eta_{\mu\nu}dx^\mu dx^\nu + dr^2 \qquad \text{where } A = \frac{r}{L}\,, \qquad (7.1)$$

which has a scaling symmetry under

$$x \to \frac{1}{\alpha}x \qquad e^A \to \alpha e^A\,. \qquad (7.2)$$

In fact this is a symmetry of all the fields in the supergravity solution and thus a symmetry of the action of a brane probing this background. In terms of the coordinates on moduli space derived in section 2, we have $u \sim e^A$ for large r and so the scaling symmetry becomes

$$x \to \frac{1}{\alpha}x \qquad u \to \alpha u\,. \qquad (7.3)$$

In other words the fields on moduli space have scaling dimension 1 which matches with the field theory prediction for the scalar components of these chiral superfields in the $\mathcal{N} = 4$ theory. Next we consider the IR end of the flow solution. Here the solution again has the scaling symmetry (7.2) except that $A = \frac{2^{5/3}r}{L}$ in this case.

The coordinate u goes like $u \sim \exp\left(\frac{r}{2^{1/3}L}\right) \sim (e^A)^{3/4}$ and thus the scaling symmetry becomes

$$x \to \frac{1}{\alpha}x \qquad u \to \alpha^{3/4}u\,. \tag{7.4}$$

Therefore, we see that the massless fields have scaling dimension 3/4 here. Again this agrees with the field theory. To see this we briefly reproduce the analysis of ref. [14, 11, 15]. The $\mathcal{N} = 4$ theory in the UV has superpotential

$$W = h\,\mathrm{Tr}([\Phi_1, \Phi_2]\Phi_3)\,, \tag{7.5}$$

and the flow under consideration corresponds to adding a mass term

$$\delta W = \frac{m}{2}\mathrm{Tr}\Phi_3^2\,. \tag{7.6}$$

The β-functions for h and m are given by

$$\begin{aligned} \beta_h &= h(d_1 + d_2 + d_3 - 3) \\ \beta_m &= m(2d_3 - 3)\,, \end{aligned} \tag{7.7}$$

where d_i is the scaling dimension of the scalar component of Φ_i. The vanishing of the β-functions (along with the SU(2) symmetry) requires $d_3 = \frac{3}{2}$, $d_1 = d_2 = \frac{3}{4}$.

A further check on the coordinates is to consider the U(1) R-symmetry which forms a part of the superconformal group. The R-charge of the scalar component of Φ_i is given by $r_i = \frac{2}{3}d_i$ and so $r_1 = r_2 = \frac{1}{2}$, $r_3 = 1$. This should match the extra U(1) symmetry of the supergravity solution in ref. [12]. To identify this U(1) symmetry we need to consider the antisymmetric tensor field (in ref. [12, eqn. 3.19])

$$C_{(2)} = e^{-i\phi}(a_1 d\theta - a_2\sigma_3 - a_3 d\phi) \wedge (\sigma_1 - i\sigma_2)\,. \tag{7.8}$$

This has a U(1) symmetry under

$$\phi \to \phi + \gamma \qquad \sigma_1 - i\sigma_2 \to e^{i\gamma}(\sigma_1 - i\sigma_2)\,. \tag{7.9}$$

The charges of the scalar fields z_1, z_2 under this symmetry indeed reproduce the field theory values.[11]

Working the other way, consider the Kähler potential at either end (UV or IR) of the flow. From the SU(2) flavour symmetry we know that K is a function of u^2

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[11] It is interesting to consider the space $z_1 = z_2 = 0$ parametrised by z_3, the vev of the massive scalar. This corresponds to the two-dimensional space with $\cos\theta = 0$. z_3 has R-charge $r_3 = 1$ and is an SU(2) scalar and therefore is of the form $z_3 = \sqrt{q}e^{-i\phi}$. We might guess that a natural choice for the radial coordinate q is to be found by putting a D-brane probe at $z_1 = z_2 = 0$ and considering its kinetic term for motions in the z_3 direction. If we put this probe metric into the form $ds^2 = g\,dz_3\,d\bar{z}_3$, for some function g, we find that $dq/dr = 2\rho^4 q/L$. At the UV fixed point $r \to \infty$, z_3 has scaling dimension 1. For the IR $r \to -\infty$, one finds that z_3 has dimension 3/2, which matches the value deduced above.

only. We also know the scaling dimension of u^2. The Kähler term in the action is of the form:

$$S = \int d^D x \; \partial_\varphi \, \partial_{\bar\varphi} K \, \partial_\mu \varphi \, \partial^\mu \bar\varphi \, , \tag{7.10}$$

where φ are the massless scalars with some scaling dimension and $D = 4$. For S to be invariant under the scaling symmetry, classically $K(u^2)$ must have scaling dimension 2. At the UV end of the flow u has scaling dimension 1, so $K \sim u^2$, as expected. At the IR end of the flow solution, u has scaling dimension 3/4 and so $K \sim (u^2)^{4/3}$. This matches the result found in section 2.5.2. It is therefore possible to recover the form of the Kähler potential at either end of the flow from a classical scaling argument.

7.2 Eleven dimensional $\mathcal{N} = 2$ flows

Now we repeat the analysis for the eleven dimension flow solution in the coordinates of section 6. First let us consider the field theory which in the UV contains four chiral superfields ϕ_i, with $i = 1 \ldots 4$. The scaling dimensions (d_i) of the fields satisfy

$$d_1 + d_2 + d_3 + d_4 = 2 \, , \tag{7.11}$$

and so $d_i = \frac{1}{2}$ by symmetry. The flow solution corresponds to perturbing the superpotential by a mass term for Φ_4

$$\delta W = \frac{m}{2} \mathrm{Tr} \Phi_4^2 \, . \tag{7.12}$$

This leads to a β-function

$$\beta_m = m(2 d_4 - 2) \, . \tag{7.13}$$

Thus the IR values of the scaling dimensions are $d_4 = 1$, $d_1 = d_2 = d_3 = \frac{1}{3}$. Again we have a U(1) R-symmetry forming part of the superconformal group and in this case the charges satisfy $r_i = d_i$.

Let us compare these results with the symmetries of the limits of the flow solution written in the coordinates of section 6. The UV end of the flow is just $AdS_4 \times S^7$ and has a symmetry under the scaling (7.2), with $A = 2r/L$. The radial coordinate on moduli space from section 6 is $\sqrt{q} \sim \exp(\frac{r}{L}) \sim (e^A)^{1/2}$ and so the scaling symmetry becomes

$$x \to \frac{1}{\alpha} x \qquad \sqrt{q} \to \alpha^{1/2} \sqrt{q} \, . \tag{7.14}$$

At the IR end of the flow $A \sim \frac{3^{3/4} r}{L}$ and $\sqrt{q} \sim \exp(\frac{r}{3^{1/4} L}) \sim (e^A)^{1/3}$. The scaling symmetry is

$$x \to \frac{1}{\alpha} x \qquad \sqrt{q} \to \alpha^{1/3} \sqrt{q} \, . \tag{7.15}$$

These are all in agreement with the scaling dimensions of the massless scalar fields derived above.

Finally we wish to match the extra U(1) symmetry of the supergravity solution to the R-symmetry of the field theory. To identify the action of the U(1) symmetry we look at the equation after (4.28) in ref. [19],

$$A^{(3)} = \frac{1}{4}\sinh\chi e^{i(-4\psi-3\phi)}(e^5 - ie^{10}) \wedge (e^6 - ie^9) \wedge (e^7 - ie^8)\,, \tag{7.16}$$

where the e^a define a frame which locally diagonalises the metric, and are listed in ref. [19]. The three form potential $A^{(3)}$ has a U(1) symmetry with

$$\psi \to \psi - \gamma \qquad \phi \to \phi + \frac{4}{3}\gamma \tag{7.17}$$

under which the coordinates w_i have charge 1/3 in agreement with the field theory.[12]

We can now consider the Kähler potential at either end of the flow, as we did at the end of section 7.1. In this case SU(3) symmetry implies that K is a function of q only. Again, we know the scaling dimension of $\sqrt{q}$. Referring to equation (7.10), but now with $n = 3$, one can see that classically K should have scaling dimension 1. At the UV end of the flow, since $\sqrt{q}$ has scaling dimension 1/2, $K \sim q$ as before. For the IR end, $\sqrt{q}$ has scaling dimension 1/3, and so K should obey $K \sim q^{3/2}$. Let us compare this with the results we found in section 6, equation (6.12). We see that for $r \to -\infty$, $K \sim e^A$ and $q \sim (e^A)^{2/3}$. This implies $K \sim q^{3/2}$, matching the result from the classical scaling argument above. Note, however, that the scaling argument does not give the Kähler potential for arbitrary r (where the scaling symmetry no longer holds).

For completeness, we note that it is possible to extract scaling dimensions for the UV and IR ends of the conifold flow of section 6.2, which provides data about the dual field theories. Since the large field limit of the flow gives the moduli space as the conifold, it is natural to assume that the dual field theory will be an orbifold, with the M2-branes at the origin. The conifold arises as the moduli space of vacua of a set of fields restricted by a D-term equation which is the defining equation (6.15) of the conifold. Then, a good description of the field theory on moduli space is in terms of fields A_i and B_j, as in ref. [29] (called X_i and Y_j in ref. [30]). There is an additional complex field, say Φ, which has a superpotential giving it a mass, which drives the flow. Carrying out the supergravity analysis as above, we find that the scaling dimensions of the A's and B's are 3/8 in the UV and 1/4 in the IR. These numbers are simply 3/4 times the dimensions of the fields in the other flow. A naive guess for the superpotential which couples all the fields is $W \sim \epsilon^{ij}\epsilon^{kl}\mathrm{Tr}(A_i B_k A_j B_l \Phi)$, for which the β-function vanishes if Φ has dimension 1/2 in the UV, and 1 in the IR, the latter also fitting nicely with a mass term $m\mathrm{Tr}\Phi^2$.

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[12] Again it interesting to consider the two-dimensional space parametrised by the vev of the massive scalar; this time it is w_4. One finds that $w_4 = \sqrt{t}e^{-i\psi}$ and so to put the probe metric into the form $ds^2 = g\, dw_3\, d\bar{w}_3$ we require $dt/dr = 2\rho^6 t/L$. It is then easy to show that the scaling dimensions at the UV or IR fixed points match those found above.

8. Closing remarks

In a previous paper [10] we studied the moduli space of a D-brane probe of a particular ten dimensional geometry dual to a flow from the $\mathcal{N} = 4$ large N SU(N) gauge theory to an $\mathcal{N} = 1$ fixed point. We found a remarkably simple form for the metric on moduli space. Asymptotically, its peculiar conical form was of particular interest, since the value of the numerical coefficients entering the metric demanded a field theory explanation.

In this paper we set out to find suitable coordinates in which to exhibit the dual supergravity geometries in order to address such questions, and we found many interesting structures. We are able to put the metric on moduli space for a brane probe into a manifestly Kähler form, which is quite natural in studies of supersymmetric gauge theory. One amusing point in these calculations has been the way in which the first order supergravity equations[13] which ensure the $\mathcal{N} = 1$ supersymmetry of the full supergravity geometry are also precisely what is needed to make the metric on moduli space Kähler.

Furthermore, we have shown that by working in these natural coordinates we can give a simple derivation of the scaling dimensions of the massless chiral superfields. The usual methods for calculating dimensions of these fields from the dual supergravity involve linearising fluctuations about a given background and is computationally rather tedious. The most direct benefit of this derivation for the purposes of this paper was the ease with which it allows us to acheive our initial goal: The Kähler metric controls all of the coefficients in the asymptotic metric, and their precise values follow from the scaling argument.

We studied many other cases, preserving various amounts of supersymmetry, and in both ten and eleven dimensions. In every case that we studied, it was possible to find an *exact* expression for the Kähler potential in terms of the supergravity fields. On the one hand, this is very exciting, since for a low amount of supersymmetry, exact results for this quantity are harder to find than for e.g., the superpotential, where holomorphicity is a powerful constraint. On the other hand, we note that in translating to the field theory we find that the exact supergravity expression for the Kähler potential does not always tell the whole story, or in the most simple way, and so we advise some caution on the part of the user. For instance, in the example of the maximal supersymmetry preserving Coulomb branch flows, the Kähler potential has a very simple form from the field theory viewpoint. However, it is looks more complicated in terms of the supergravity scalars. One has to unpack the details of the supergravity fields and/or change variables to another set of coordinates in which the simplicity is more manifest [21, 22, 6, 25], to see that in those cases there is in fact a simpler story to be told. Another example comes from the two eleven

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[13] These are derived from Killing spinor equations (see e.g. ref. [11]) or a Bogomol'nyi-type argument (see ref. [22] and references therein).

dimensional flow geometries of section 6. One simple expression for the supergravity Kähler potential describes two different theories, the details of which must still be unpacked by using the knowledge of the detailed behaviour of the scalars.

In each of the examples we have considered, the Kähler potential satisfies a remarkable simple differential equation (2.34) (or (4.11), (4.12) in eleven dimensions), which we conjecture to be universal for these sorts of holographic RG flows (we proved them to be so for all the Coulomb branch flows). It will be interesting to try to understand better the origins of this equation in the gauged supergravity and also its direct interpretation in the dual gauge theory.

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